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CLASS XII — MATHEMATICS

Chapter 1: Relations and Functions

Complete Study Notes with Solved Examples, Exercise-wise Concept Summary, Additional Practice Questions & Solutions

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What this chapter covers

- Types of relations: empty, universal, reflexive, symmetric, transitive, and equivalence relations
- Equivalence classes and how an equivalence relation partitions a set
- Types of functions: one-one (injective), onto (surjective), and bijective functions
- Composition of functions and invertible functions
- A key finite-vs-infinite-set distinction: for finite sets, one-one $\Leftrightarrow$ onto (for self-maps); this fails for infinite sets

1. Types of Relations

1.1 Core Idea

A relation R from set A to set B is mathematically defined as an arbitrary subset of A × B. If (a, b) ∈ R, we say 'a is related to b under R' and write a R b.

A relation in a set A (i.e. from A to itself) is a subset of A × A.

1.2 Trivial Relations

Definition 1 (Empty relation): $R = \phi \subset A \times A$ — no element of A is related to any element of A.

Definition 2 (Universal relation): $R = A \times A$ — every element of A is related to every element of A.

Both are called trivial relations.

Example 1: Boys' school relations

R = {(a,b) : a is sister of b} — since it's a boys' school, no student can be anyone's sister, so R = ϕ (empty relation).

R′ = {(a,b) : height difference between a and b is less than 3 m} — true for every pair, so R′ = A × A (universal relation).

1.3 Reflexive, Symmetric, Transitive Relations

Definition 3: A relation R in set A is:
(i) Reflexive if $(a,a) \in R$ for every $a \in A$
(ii) Symmetric if $(a_1,a_2) \in R \Rightarrow (a_2,a_1) \in R$
(iii) Transitive if $(a_1,a_2) \in R$ and $(a_2,a_3) \in R \Rightarrow (a_1,a_3) \in R$

Definition 4 (Equivalence relation): R is an equivalence relation if it is reflexive, symmetric AND transitive.

1.4 Worked Examples from the NCERT Text

Example 2: Congruence of triangles is an equivalence relation

Reflexive: every triangle is congruent to itself.

Symmetric: $T_1 \cong T_2 \Rightarrow T_2 \cong T_1$.

Transitive: $T_1 \cong T_2$ and $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$. Hence an equivalence relation.

Example 3: Perpendicularity of lines — symmetric but NOT reflexive or transitive

Not reflexive: a line is never perpendicular to itself.

Symmetric: $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$.

Not transitive: if $L_1 \perp L_2$ and $L_2 \perp L_3$, then L_1 is actually PARALLEL to L_3, not perpendicular.

Example 4: R = {(1,1),(2,2),(3,3),(1,2),(2,3)} on {1,2,3} — reflexive but NOT symmetric or transitive

Reflexive: (1,1),(2,2),(3,3) all present.

Not symmetric: $(1,2) \in R$ but $(2,1) \notin R$.

Not transitive: $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$.

Example 5: 'R = {(a,b) : 2 divides a−b}' on Z is an equivalence relation

Reflexive: 2 divides $(a - a) = 0$ for all a.

Symmetric: if $2 \mid (a-b)$ then $2 \mid (b-a)$ (just the negative).

Transitive: if $2|(a-b)$ and $2|(b-c)$, then $a-c = (a-b)+(b-c)$ is even, so $2|(a-c)$.

1.5 Equivalence Classes — Key Concept

An equivalence relation R on set X partitions X into mutually disjoint subsets A_i (equivalence classes) such that: all elements within an A_i are related to each other, no element of A_i is related to any element of A_j ($i \neq j$), and $\cup A_i = X$.

In Example 5 above: the set of even integers E = [0] and the set of odd integers O = [1] are the two equivalence classes; $Z = E \cup O$ and $E \cap O = \phi$.

Example 6: R = {(a,b) : both a,b odd or both even} on {1,...,7}

This is an equivalence relation (same reasoning pattern as Example 5).

{1,3,5,7} (all odd) are mutually related; {2,4,6} (all even) are mutually related.

No element of {1,3,5,7} is related to any element of {2,4,6} — two separate equivalence classes partition the set.

1.6 Exercise 1.1 — Concept Notes

- Always check ALL THREE properties (reflexive, symmetric, transitive) independently — a relation can have any combination of these being true/false.
- To DISPROVE a property, one single counter-example pair is enough (e.g. find one $(a,b) \in R$ with $(b,a) \notin R$ to disprove symmetry).
- To PROVE a property holds in general, you must argue for an arbitrary element/pair, not just check specific cases.
- A quick reflexivity check: does $(a,a) \in R$ make sense for the given condition? E.g. '$a \leq b^3$' — reflexive requires $a \leq a^3$, which fails for $0 < a < 1$, so NOT reflexive.
- For equivalence relations on finite sets, the relation effectively partitions the set into equivalence classes — a powerful way to verify or construct such relations.

Additional Practice Questions — Exercise 1.1

Q. 1. Check whether R = {(a,b) : a and b have the same birthday} defined on the set of all people is an equivalence relation.

Reflexive: everyone shares their own birthday with themselves. ✓

Symmetric: if a shares a's birthday with b, then b shares it with a. ✓

Transitive: if a,b share a birthday and b,c share a birthday, then a,c share that birthday too. ✓

R is reflexive, symmetric and transitive, hence an equivalence relation.

Q. 2. Show that R = {(a,b) : $a \geq b$} on R (real numbers) is reflexive and transitive but not symmetric.

Reflexive: $a \geq a$ is always true. ✓

Transitive: $a \geq b$ and $b \geq c \Rightarrow a \geq c$. ✓

Not symmetric: $5 \geq 3$ is true but $3 \geq 5$ is false, so $(5,3) \in R$ but $(3,5) \notin R$.

Q. 3. Let R be the relation on set {1,2,3,4,5} given by R = {(a,b) : $|a-b| < 3$}. Is R reflexive? symmetric?

Reflexive: $|a-a| = 0 < 3$ always true. ✓Reflexive.

Symmetric: $|a-b| = |b-a|$ always, so $(a,b) \in R \Rightarrow (b,a) \in R$. ✓Symmetric.

Not transitive: $(1,3) \in R$ since $|1-3|=2<3$, $(3,5) \in R$ since $|3-5|=2<3$, but $(1,5)$: $|1-5|=4$, not < 3, so $(1,5) \notin R$.

Q. 4. Show that the relation R on the set of all triangles defined by R = {(T1,T2): T1 has the same area as T2} is an equivalence relation.

Reflexive: any triangle has the same area as itself.

Symmetric: if T1's area equals T2's area, then T2's area equals T1's.

Transitive: if area(T1)=area(T2) and area(T2)=area(T3), then area(T1)=area(T3).

Hence R is an equivalence relation (note: equal-area triangles need not be congruent or similar).

2. Types of Functions

2.1 One-One (Injective) Functions

Definition 5: f : X → Y is one-one (injective) if distinct elements of X map to distinct elements of Y, i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Otherwise, f is many-one.

2.2 Onto (Surjective) Functions

Definition 6: f : X → Y is onto (surjective) if every element of Y is the image of some element of X, i.e. for every $y \in Y$, there exists $x \in X$ such that f(x) = y.

Remark: f : X → Y is onto if and only if Range of f = Y (the co-domain).

2.3 Bijective Functions

Definition 7: f : X → Y is one-one and onto (bijective) if f is both one-one and onto.

2.4 Worked Examples from the NCERT Text

Example 7: Roll-number function is one-one but not onto

f : A(50 students) → N, f(x) = roll number of x. No two students share a roll number → one-one.

Not onto: e.g. $51 \in N$ is not any student's roll number, so it's never in the range.

Example 8: f : N → N, f(x) = 2x — one-one but not onto

One-one: $2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Not onto: $1 \in N$ has no pre-image since 2x = 1 has no natural number solution.

Example 9: f : R → R, f(x) = 2x — one-one AND onto

One-one: same as above. Onto: for any $y \in R$, $x = y/2 \in R$ satisfies f(x) = y.

(Contrast with Example 8 — same rule, but different domain/co-domain changes onto-ness!)

Example 10: f(1)=f(2)=1, f(x)=x−1 for x>2 — onto but NOT one-one

Not one-one: f(1) = f(2) = 1.

Onto: every $y \neq 1$ has pre-image y+1; and y=1 has pre-image 1 (or 2).

Example 11: f : R → R, f(x) = x^2 — neither one-one nor onto

Not one-one: f(−1) = 1 = f(1).

Not onto: −2 has no real square root, so it's never in the range.

Example 13 & 14 — Important theorem for FINITE sets

An onto function f : {1,2,3} → {1,2,3} is always one-one, AND a one-one function f : {1,2,3} → {1,2,3} is always onto.

This holds for ANY finite set X: for f : X → X, one-one $\Leftrightarrow$ onto. This is a defining characteristic of finite sets — it FAILS for infinite sets (see Examples 8 and 10 above, both f : N → N).

2.5 Exercise 1.2 — Concept Notes

- Standard method to check ONE-ONE: assume $f(x_1) = f(x_2)$, algebraically derive whether this forces $x_1 = x_2$.

- Standard method to check ONTO: take arbitrary y in the co-domain, try to solve f(x) = y for x within the stated domain — if you always can, it's onto; if some y has no valid pre-image, it's not.
- The SAME formula (e.g. $f(x)=x^2$, $f(x)=x^3$) can behave completely differently depending on domain/co-domain (N vs Z vs R) — always check the specific sets given, don't assume from the formula alone.
- x^3 is one-one AND onto on R (odd powers preserve sign & are strictly increasing), but x^2 is neither one-one nor onto on R, though it IS one-one (but not onto) if domain is restricted to non-negative reals only.
- The Greatest Integer Function [x], Modulus Function |x|, and Signum Function are all standard 'neither one-one nor onto' examples — know their reasoning by heart.
- For piecewise-defined functions, check across ALL cases/branches (odd vs even inputs etc.) that no two branches accidentally produce equal outputs.

Additional Practice Questions — Exercise 1.2

Q. 1. Show that f : R → R defined by f(x) = 5x + 3 is one-one and onto.

One-one: $f(x_1)=f(x_2) \Rightarrow 5x_1+3 = 5x_2+3 \Rightarrow x_1 = x_2$.

Onto: for any $y \in R$, $x = (y-3)/5 \in R$ satisfies f(x) = y. Hence bijective.

Q. 2. Check whether f : Z → Z defined by f(x) = 2x + 1 is one-one and onto.

One-one: $2x_1+1 = 2x_2+1 \Rightarrow x_1 = x_2$. ✓

Onto: for $y = 2 \in Z$ (co-domain), we'd need $x = (2-1)/2 = 0.5$, which is NOT an integer. So f is not onto.

Conclusion: f is one-one but not onto.

Q. 3. Show that f : N → N given by $f(x) = x^2 + x + 1$ is one-one but not onto.

One-one: f is strictly increasing for $x \in N$ (since both x^2 and x increase), so $f(x_1)=f(x_2)$ forces $x_1=x_2$.

Not onto: f(1) = 3 is the smallest value in the range, so 1 and 2 in N (co-domain) have no pre-image.

Q. 4. Let A = {1,2,3}. How many bijective functions exist from A to A?

A bijective function from a finite set to itself is a permutation of its elements.

Number of permutations of 3 elements = 3! = 6.

3. Composition of Functions and Invertible Functions

3.1 Composition of Functions

Definition 8: Let f : A → B and g : B → C. The composition gof : A → C is defined by gof(x) = g(f(x)) for all x ∈ A.

Example 15: Finding gof for finite sets

f(2)=3,f(3)=4,f(4)=f(5)=5; g(3)=g(4)=7, g(5)=g(9)=11

gof(2)=g(3)=7, gof(3)=g(4)=7, gof(4)=g(5)=11, gof(5)=g(5)=11

Example 16: gof ≠ fog in general

f(x)=cos x, g(x)=3x^2. gof(x) = 3$\cos^2$x. fog(x) = cos(3x^2).

At x=0: gof(0) = $3(1)^2$ = 3, fog(0) = cos(0) = 1. Since 3 ≠ 1, gof ≠ fog — composition is NOT commutative in general.

3.2 Invertible Functions

Definition 9: f : X → Y is invertible if there exists g : Y → X such that gof = I_x (identity on X) and fog = I_Y (identity on Y). Such g is called f^{-1}, the inverse of f.

KEY THEOREM: f is invertible ⇔ f is BOTH one-one and onto (bijective). This lets us prove invertibility just by checking bijectivity, without needing to construct the actual inverse.

Example 17: f : N → Y, f(x) = 4x+3, where Y = {y ∈ N : y = 4x+3 for some x ∈ N}

Given y ∈ Y, y = 4x+3 ⇒ x = (y−3)/4. Define g(y) = (y−3)/4.

gof(x) = g(4x+3) = (4x+3−3)/4 = x = I□(x)

fog(y) = f((y−3)/4) = 4·(y−3)/4 + 3 = y = I_Y(y)

Since gof = I□ and fog = I_Y, f is invertible with inverse g(y) = (y−3)/4.

3.3 Exercise Notes (Composition & Invertibility) — Concept Summary

- To find gof(x): first apply the INNER function f, then apply g to that result. Order matters — gof means 'g of f', apply f first.
- To prove invertibility WITHOUT finding the inverse explicitly: just show the function is one-one and onto (bijective). This is often much faster than constructing g directly.
- To find the inverse EXPLICITLY: set y = f(x), solve algebraically for x in terms of y — that expression (with x, y swapped in labeling) is f^{-1}(y).
- Always verify BOTH gof = I_x and fog = I_Y when asked to 'verify' an inverse — checking only one direction is insufficient in general (though for functions between the same set it often coincides).

Additional Practice Questions — Composition & Invertible Functions

Q. 1. If f(x) = x^2 and g(x) = x+3 (both R → R), find gof and fog. Are they equal?

gof(x) = g(f(x)) = g(x^2) = x^2 + 3

fog(x) = f(g(x)) = f(x+3) = $(x+3)^2$= x^2 + 6x + 9

gof ≠ fog since x^2+3 ≠ x^2+6x+9 in general (e.g. at x=1: 4 ≠ 16).

Q. 2. Show that f : R → R given by f(x) = 7x − 4 is invertible, and find f^{-1}.

One-one: $7x_1-4 = 7x_2-4 \Rightarrow x_1=x_2$. Onto: for any $y \in R$, $x = (y+4)/7 \in R$ gives $f(x)=y$.

Since f is bijective, it is invertible.

Let $y = 7x-4 \Rightarrow x = (y+4)/7$. So $f^{-1}(y) = (y+4)/7$ (or equivalently $f^{-1}(x) = (x+4)/7$).

Q. 3. If $f : R \to R$ is $f(x) = x^3$ and $g : R \to R$ is $g(x) = 2x-1$, verify $(gof)(2)$ using both direct computation and the definition.

$f(2) = 2^3 = 8$. $gof(2) = g(f(2)) = g(8) = 2(8)-1 = 15$.

Direct formula: $gof(x) = g(x^3) = 2x^3-1 \to gof(2) = 2(8)-1 = 15$. Matches ✓

4. Miscellaneous Exercise — Concept Notes

This section blends relations and functions into more abstract, proof-style problems: showing intersections of equivalence relations are equivalence relations, counting relations/functions with given properties, and working with equal functions.

Example 18: Intersection of two equivalence relations is an equivalence relation

If R_1, R_2 are both equivalence relations on A, then $R_1 \cap R_2$ is reflexive (both contain (a,a)), symmetric (both preserve symmetry, so intersection does too), and transitive (similarly) — hence also an equivalence relation.

General principle: reflexivity, symmetry and transitivity are all preserved under intersection of relations (but NOT necessarily under union).

Example 21: R = {(a,b) : f(a) = f(b)} for any function f is always an equivalence relation

Reflexive: f(a)=f(a). Symmetric: f(a)=f(b) ⇒f(b)=f(a). Transitive: f(a)=f(b), f(b)=f(c) ⇒f(a)=f(c).

Useful general fact: 'having the same image under any function' is ALWAYS an equivalence relation — a common exam trick.

Example 22: Number of one-one functions from {1,2,3} to itself = 3! = 6

A one-one map from a finite set to itself is just a permutation. General formula: n! one-one (= onto = bijective) self-maps of an n-element set.

Examples 23 & 24: Counting relations with specific properties

Example 23: relations on {1,2,3} containing (1,2) and (2,3), reflexive & transitive but NOT symmetric — answer is 3.

Example 24: equivalence relations on {1,2,3} containing (1,2) and (2,1) — answer is 2 (the minimal one, and the universal relation).

Technique: start from the smallest relation forced by reflexivity + the given pairs, then see what MUST be added to preserve transitivity/symmetry, and whether that forces the relation to become the universal relation.

4.1 Exercise Notes for Miscellaneous Exercise

- 'Show X is an equivalence relation' proofs in this section often reduce to checking reflexive/symmetric/transitive using algebraic manipulation of an equation (e.g. xv = yu type conditions) — the technique from Exercise 1.1 carries over directly.
- Counting-type questions (number of relations/equivalence relations satisfying constraints) require systematically building up the minimal required relation, then checking what forced additions occur — very common in exams and best practiced repeatedly.
- 'Are f and g equal functions?' questions require checking f(a) = g(a) for EVERY a in the domain, not just a formula match — two different-looking formulas can define the same function over a specific finite domain.
- Number of onto functions from an n-element set to itself = n! (same as one-one, by the finite-set theorem).

Additional Practice Questions — Miscellaneous Exercise

Q. 1. Let A = {1, 2, 3} and B = {4, 5}. Find the number of onto functions from A to B.

Total functions from A to B = 2^3 = 8.

Functions that are NOT onto: all map to only {4} (1 way) or only {5} (1 way) = 2 functions.

Onto functions = 8 − 2 = 6.

Q. 2. If R_1 and R_2 are symmetric relations on a set A, show that $R_1 \cup R_2$ is also symmetric (but note this need not hold for transitivity).

Let $(a,b) \in R_1 \cup R_2$. Then $(a,b) \in R_1$ or $(a,b) \in R_2$.

If $(a,b) \in R_1$, since R_1 symmetric, $(b,a) \in R_1 \subset R_1 \cup R_2$. Similarly if $(a,b) \in R_2$.

Either way $(b,a) \in R_1 \cup R_2$, so $R_1 \cup R_2$ is symmetric. (Transitivity, however, is generally NOT preserved under union — this is why the miscellaneous exercise emphasizes intersection instead.)

Q. 3. Let f, g : {1,2,3} → {1,2,3} be given by f = {(1,2),(2,3),(3,1)} and g = {(1,3),(2,1),(3,2)}. Find gof and fog.

gof(1) = g(f(1)) = g(2) = 1. gof(2) = g(3) = 2. gof(3) = g(1) = 3.

So gof = {(1,1),(2,2),(3,3)} = Identity function.

fog(1) = f(g(1)) = f(3) = 1. fog(2) = f(1) = 2. fog(3) = f(2) = 3.

So fog = {(1,1),(2,2),(3,3)} = Identity function too. Here gof = fog = I (g is in fact f^{-1}).

5. Chapter Summary & Quick-Reference Sheet

5.1 Formula / Definition Sheet

Concept	Definition / Formula
Empty relation	$R = \phi \subset X \times X$
Universal relation	$R = X \times X$
Reflexive	$(a,a) \in R$ for all $a \in X$
Symmetric	$(a,b) \in R \Rightarrow (b,a) \in R$
Transitive	$(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
Equivalence relation	Reflexive + Symmetric + Transitive
One-one (injective)	$f(x_1)=f(x_2) \Rightarrow x_1=x_2$
Onto (surjective)	Range of f = Co-domain
Bijective	One-one AND onto
Composition	$gof(x) = g(f(x))$
Invertible	f bijective $\Leftrightarrow$ f invertible; $gof=I_x$, fog=I_Y
Finite set self-map theorem	$f:X\rightarrow X$, X finite: one-one $\Leftrightarrow$ onto
One-one/onto self-maps count	n! for an n-element set

5.2 Common Exam Pitfalls

- Assuming a function's one-one/onto behavior from its FORMULA alone — always check the stated domain and co-domain (N, Z, R all behave differently for the same rule).
- Forgetting to check ALL of reflexive/symmetric/transitive independently — they are logically unrelated properties.
- Confusing gof with fog — remember gof means apply f FIRST, then g (read right to left).
- Believing 'one-one $\Leftrightarrow$ onto' works for infinite sets or across different domain/co-domain sizes — this equivalence ONLY holds for self-maps ($f:X\rightarrow X$) on FINITE sets.
- Trying to prove invertibility by guessing the inverse formula first — it's often faster to prove bijectivity abstractly, especially when the exact algebraic inverse is hard to isolate.
- In counting-relations problems, forgetting that adding one pair for symmetry/transitivity often forces several more pairs — always trace the full chain of forced additions.

— End of Chapter 1 Notes —